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On the Relation between Some Important Notions of Projective and Metrical Differential Geometry.[*]

BY F. M. MORRISON.

I. INTRODUCTION.

In his papers on the "Projective Differential Geometry of Curved Surfaces"[†] Professor E. J. Wilczynski has shown that the projective differential geometry of a surface may be based upon the consideration of a completely integrable system of two linear partial differential equations of the second order, which may be reduced to what he calls the intermediate form

$$y_{uu}+2ay_u+2by_v+cy=0, \quad y_{vv}+2a'y_u+2b'y_v+c'y=0. \tag{1}$$

Such a system of equations has just four linearly independent solutions, $y', y'', y^{(3)}, y^{(4)}$. If these be taken as the homogeneous coordinates of a point P_y in space, as u and v assume all of their values, the point P_y will describe a surface S, an integral surface of equations (1). This surface will be non-degenerate and non-developable, and will have the curves $u=$const. and $v=$ const. as asymptotic lines. The most general integral surface of equations (1) is a projective transformation of any particular one, so that the coefficients of equations (1), in so far as they are intrinsically connected with the surface at all, will involve only its projective properties. But these coefficients also depend upon certain accidental elements of the analytic representation of the surface. To eliminate these accidental elements the notions of invariants and covariants are introduced.[‡] The projective properties of the surface are then expressed by invariant equations or system of equations. Other surfaces and other geometrical configurations which have a projective relation to it will be given by the covariants.

The four fundamental semi-covariants of system (1) are

$$y, \quad z=y_u+ay, \quad \rho=y_v+b'y, \quad \sigma=y_{uv}+b'y_u+ay_v+\tfrac{1}{2}(a_v+b'_u+2ab')y. \tag{2}$$

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[†] *Transactions of the American Mathematical Society,* Vols. VIII, IX and X.

[‡] *Transactions,* Vol. VIII, p. 241, *et seq.*

If four linearly independent solutions y', y'', $y^{(3)}$, $y^{(4)}$ of (1) be put for y in the semi-covariants (2), three points P_z, P_ρ, P_σ will be obtained which are semi-covariantly connected with the point P_y.

By a properly chosen transformation of the form $y=\lambda\bar{y}$ equations (1) may be changed into the canonical form

$$y_{uu}+2by_v+fy=0, \quad y_{vv}+2a'y_u+gy=0. \tag{3}$$

The semi-covariants of (1) and (3) are connected with each other by the equations $y=\lambda\bar{y}$, $z=\lambda\bar{z}$, $\rho=\lambda\bar{\rho}$, $\sigma=\lambda\bar{\sigma}$. The semi-covariant points are the same for both (1) and (3). These points P_y, P_z, P_ρ, P_σ form in general a non-degenerate tetrahedron which Professor Wilczynski has systematically used as tetrahedron of reference, for the purpose of studying the properties of a surface in the neighborhood of any one of its points.*

If the Cartesian coordinates of the surface are regarded as known, we may identify them with y', y'', $y^{(3)}$ and make $y^{(4)}$ equal unity. In the resulting system of partial differential equations of form (1) we shall then have $c=c'=0$, and system (1) becomes

$$y_{uu}+2ay_u+2by_v=0, \quad y_{vv}+2a'y_u+2b'y_v=0. \tag{4}$$

Let us now consider the familiar Gauss' equations of the metrical theory of surfaces, viz.:

$$\left.\begin{aligned} x_{uu} &= \begin{Bmatrix}11\\1\end{Bmatrix}x_u+\begin{Bmatrix}11\\2\end{Bmatrix}x_v+DX, \quad x_{uv}=\begin{Bmatrix}12\\1\end{Bmatrix}x_u+\begin{Bmatrix}12\\2\end{Bmatrix}x_v+D'X,\\ x_{vv} &= \begin{Bmatrix}22\\1\end{Bmatrix}x_u+\begin{Bmatrix}22\\2\end{Bmatrix}x_v+D''X. \end{aligned}\right\} \tag{5}$$

If D and D'' are equal to zero, the curves $u=$const. and $v=$const. are asymptotic lines, and we observe that the first and third of these equations are now of exactly the same form as equations (4). We can therefore express directly the coefficients of equations (4) in terms of the Christoffel symbols occurring in (5). We shall then be able to change from the homogeneous coordinate system with a semi-covariant tetrahedron of reference to any suitable Cartesian system. The coefficients of these transformations will be functions of the quantities E, F, G and D' and their derivatives with respect to u and v, quantities which are fundamental in the metrical theory of surfaces.

In this paper we shall determine these transformations and study some of the metrical properties of certain geometrical configurations associated with a point on a surface, which have been defined and investigated from a projective point of view by Professor Wilczynski, but which have not as yet been studied metrically. We can define by means of these configurations some new classes of special points on a surface and some new kinds of surfaces. We shall also apply these notions to a well-known special surface, the Minimal Surface of Enneper.

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* *Transactions*, Vol. IX, p. 79.

II. THE FUNDAMENTAL TRANSFORMATIONS OF COORDINATES.

The integrability conditions of system (4) are

$$\begin{aligned} &a_v - b'_u = 0, \\ &a'_{uu} - 2a'a_u - 2aa'_u - (a_{vv} + 2b'a_v - 2ba'_v - 4a'b_v) = 0, \\ &b_{vv} - 2bb'_v - 2b'b_v - (b'_{uu} + 2ab'_u - 2a'b_u - 4ba'_u) = 0. \end{aligned}$$

We shall assume that they are satisfied, so that (4) has four linearly independent solutions y', y'', $y^{(3)}$, 1, which we shall identify with the Cartesian coordinates of a point of a surface.

Comparing equations (4) and the first and third of equations (5) we have the relations

$$a = -\tfrac{1}{2}\left\{\begin{matrix} 1\,1 \\ 1 \end{matrix}\right\}, \quad a' = -\tfrac{1}{2}\left\{\begin{matrix} 2\,2 \\ 1 \end{matrix}\right\}, \quad b = -\tfrac{1}{2}\left\{\begin{matrix} 1\,1 \\ 2 \end{matrix}\right\}, \quad b' = -\tfrac{1}{2}\left\{\begin{matrix} 2\,2 \\ 2 \end{matrix}\right\}, \qquad (6)$$

which express the values of the quantities a, b, a', b' in terms of the Christoffel symbols of the classical theory.

The *homogeneous* Cartesian coordinates of the points P_y, P_z, P_ρ and P_σ are obtained by substituting in order y', y'', $y^{(3)}$, $y^{(4)} = 1$ in each of the expressions (2). The *non-homogeneous* Cartesian coordinates of each of these points are the ratios of the corresponding four homogeneous coordinates concerned. The values obtained by this method for the Cartesian coordinates of the four points P_y, P_z, P_ρ, P_σ are given in the following table:

$$\begin{aligned} &P_y\colon\ y',\ y'',\ y^{(3)}; \qquad P_z\colon\ y' + y'_u/a,\ y'' + y''_u/a,\ y^{(3)} + y^{(3)}_u/a; \\ &P_\rho\colon\ y' + y'_v/b',\ y'' + y''_v/b',\ y^{(3)} + y^{(3)}_v/b'; \\ &P_\sigma\colon\ y' + (b'y'_u + ay'_v + y'_{uv})/R,\ y'' + (b'y''_u + ay''_v + y''_{uv})/R, \\ &\qquad\qquad\qquad\qquad\qquad\qquad y^{(3)} + (b'y^{(3)}_u + ay^{(3)}_v + y^{(3)}_{uv})/R, \end{aligned}$$

where

$$R = \tfrac{1}{2}(a_v + b'_u + 2ab').$$

In speaking of y', y'', $y^{(3)}$ as the Cartesian coordinates of a point of the surface, we imply that some Cartesian system of coordinates, fixed in space, has been chosen as system of reference. Referred to this fixed Cartesian system, let the coordinates of a general point P of space be $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ and let the coordinates of the same point with reference to the semi-covariant surface tetrahedron P_y, P_z, P_ρ, P_σ be x_1, x_2, x_3, x_4. Then the relations between the two sets of coordinates of the point P are*

$$\left.\begin{aligned}
\omega\bar{\xi} &= x_1 y' + x_2(y'_u + a y') + x_3(y'_v + b' y') + x_4(y'_{uv} + b' y'_u + a y'_v + R y'),\\
\omega\bar{\eta} &= x_1 y'' + x_2(y''_u + a y'') + x_3(y''_v + b' y'') + x_4(y''_{uv} + b' y''_u + a y''_v + R y''),\\
\omega\bar{\zeta} &= x_1 y^{(3)} + x_2(y^{(3)}_u + a y^{(3)}) + x_3(y^{(3)}_v + b' y^{(3)}) + x_4(y^{(3)}_{uv} + b' y^{(3)}_u + a y^{(3)}_v + R y^{(3)}),\\
\omega &= x_1 + x_2 a + x_3 b' + x_4 R,
\end{aligned}\right\} \quad (7)$$

where we have written these relations in homogeneous form by introducing a factor of proportionality, ω.

The determinant of the right members of equations (7), which will be designated by Δ, is different from zero for any non-developable surface. As we shall see presently, it is equal to $-HD'$ (see equations (10)).

Let us write $y'=x$, $y''=y$, $y^{(3)}=z$ in agreement with the usual notation for the Cartesian coordinates of a point. Then equations (7) become

$$\left.\begin{aligned}
\omega\bar{\xi} &= x_1 x + x_2(x_u + a x) + x_3(x_v + b' x) + x_4(x_{uv} + b' x_u + a x_v + R x),\\
\omega\bar{\eta} &= x_1 y + x_2(y_u + a y) + x_3(y_v + b' y) + x_4(y_{uv} + b' y_u + a y_v + R y),\\
\omega\bar{\zeta} &= x_1 z + x_2(z_u + a z) + x_3(z_v + b' z) + x_4(z_{uv} + b' z_u + a z_v + R z),\\
\omega &= x_1 + x_2 a + x_3 b' + x_4 R.
\end{aligned}\right\} \quad (8)$$

If there be substituted in equations (8) the values of x_{uv}, y_{uv}, z_{uv} as given by the second Gauss equation

$$x_{uv} = \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix} x_u + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix} x_v + D'X,$$

which holds also if y, z and Y, Z be substituted for x and X, respectively, these equations become

$$\left.\begin{aligned}
\omega\bar{\xi} &= x_1 x + x_2(x_u + a x) + x_3(x_v + b' x)\\
&\qquad + x_4[(b' - 2c)x_u + (a - 2d)x_v + R x + D'X],\\
\omega\bar{\eta} &= x_1 y + x_2(y_u + a y) + x_3(y_v + b' y)\\
&\qquad + x_4[(b' - 2c)y_u + (a - 2d)y_v + R y + D'Y],\\
\omega\bar{\zeta} &= x_1 z + x_2(z_u + a z) + x_3(z_v + b' z)\\
&\qquad + x_4[(b' - 2c)z_u + (a - 2d)z_v + R z + D'Z],\\
\omega &= x_1 + x_2 a + x_3 b' + x_4 R,
\end{aligned}\right\} \quad (9)$$

where

$$c = -\tfrac{1}{2}\begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}, \quad d = -\tfrac{1}{2}\begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}, \quad \Delta = -HD', \qquad (10)$$

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* *Transactions*, Vol. IX, p. 80.

and where X, Y, Z, H are the quantities usually denoted by these letters in the metrical theory of surfaces.

Solving equations (9) for x_1, x_2, x_3, x_4, we find

$$\left.\begin{aligned}
\omega x_1 = {} & \{SH^2X+D'[(aG-b'F)x_u-(aF-b'E)x_v]\}(\bar{\xi}-x) \\
& +\{SH^2Y+D'[(aG-b'F)y_u-(aF-b'E)y_v]\}(\bar{\eta}-y) \\
& +\{SH^2Z+D'[(aG-b'F)z_u-(aF-b'E)z_v]\}(\bar{\zeta}-z)-H^2D', \\
\omega x_2 = {} & \{(b'-2c)H^2X-D'(Gx_u-Fx_v)\}(\bar{\xi}-x) \\
& +\{(b'-2c)H^2Y-D'(Gy_u-Fy_v)\}(\bar{\eta}-y) \\
& +\{(b'-2c)H^2Z-D'(Gz_u-Fz_v)\}(\bar{\zeta}-z), \\
\omega x_3 = {} & \{(a-2d)H^2X+D'(Fx_u-Ex_v)\}(\bar{\xi}-x) \\
& +\{(a-2d)H^2Y+D'(Fy_u-Ey_v)\}(\bar{\eta}-y) \\
& +\{(a-2d)H^2Z+D'(Fz_u-Ez_v)\}(\bar{\zeta}-z), \\
\omega x_4 = {} & -H^2X(\bar{\xi}-x)-H^2Y(\bar{\eta}-y)-H^2Z(\bar{\zeta}-z),
\end{aligned}\right\} \quad (11)$$

where we have put

$$S=\tfrac{1}{2}(a_v+b'_u-2ab')+2(ac+bd).$$

In equations (9) and (11) we have transformations of the kind desired for the purposes of this paper, but we shall find it of advantage to make use also of the moving Cartesian system made up of the surface normal and the lines of curvature tangents of a general point on the surface. We shall speak of this system of coordinate axes as the *surface trihedral.*

The direction cosines of the tangents to the asymptotic curves $v=$const. and $u=$const. are

$$x_u/\sqrt{E},\ y_u/\sqrt{E},\ z_u/\sqrt{E} \quad \text{and} \quad x_v/\sqrt{G},\ y_v/\sqrt{G},\ z_v/\sqrt{G},$$

respectively. Therefore, the direction cosines of the tangents to the lines of curvature are

$$(x_v\sqrt{E}+x_u\sqrt{G})/2\beta\sqrt{EG},\quad (y_v\sqrt{E}+y_u\sqrt{G})/2\beta\sqrt{EG},\quad (z_v\sqrt{E}+z_u\sqrt{G})/2\beta\sqrt{EG}$$

and

$$(x_v\sqrt{E}-x_u\sqrt{G})/2\alpha\sqrt{EG},\quad (y_v\sqrt{E}-y_u\sqrt{G})/2\alpha\sqrt{EG},\quad (z_v\sqrt{E}-z_u\sqrt{G})/2\alpha\sqrt{EG},$$

where

$$\alpha=\sqrt{(\sqrt{EG}-F)/2\sqrt{EG}},\quad \beta=\sqrt{(\sqrt{EG}+F)/2\sqrt{EG}}. \qquad (12)$$

Let the coordinates of any point referred to the surface trihedral be ξ, η, ζ. We select the positive directions of these axes in such a way that the transformation of coordinates from the fixed Cartesian system to the surface trihedral is given by the following equations:

$$\left.\begin{aligned}
\bar{\xi} &= \frac{x_v\sqrt{E}+x_u\sqrt{G}}{2\beta\sqrt{EG}}\xi+\frac{x_v\sqrt{E}-x_u\sqrt{G}}{2\alpha\sqrt{EG}}\eta+X\zeta+x,\\
\bar{\eta} &= \frac{y_v\sqrt{E}+y_u\sqrt{G}}{2\beta\sqrt{EG}}\xi+\frac{y_v\sqrt{E}-y_u\sqrt{G}}{2\alpha\sqrt{EG}}\eta+Y\zeta+y,\\
\bar{\zeta} &= \frac{z_v\sqrt{E}+z_u\sqrt{G}}{2\beta\sqrt{EG}}\xi+\frac{z_v\sqrt{E}-z_u\sqrt{G}}{2\alpha\sqrt{EG}}\eta+Z\zeta+z.
\end{aligned}\right\} \tag{13}$$

Applying equations (13) to (11) we obtain

$$\left.\begin{aligned}
\omega x_1 &= \alpha D'(a\sqrt{G}+b'\sqrt{E})\xi-\beta D'(a\sqrt{G}-b'\sqrt{E})\eta+HS\zeta-HD',\\
\omega x_2 &= -\alpha D'\sqrt{G}\xi+\beta D'\sqrt{G}\eta+H(b'-2c)\zeta,\\
\omega x_3 &= -\alpha D'\sqrt{E}\xi-\beta D'\sqrt{E}\eta+H(a-2d)\zeta,\\
\omega x_4 &= -H\zeta.
\end{aligned}\right\} \tag{14}$$

These equations express the relations between the homogeneous coordinates x_1, x_2, x_3, x_4 *of a point, referred to the surface tetrahedron* P_y, P_z, P_ρ, P_σ *and the Cartesian coordinates* ξ, η, ζ *of the same point, referred to the surface trihedral composed of the surface normal and the lines of curvature tangents.*

III. THE OSCULATING LINEAR COMPLEXES OF THE ASYMPTOTIC CURVES.

The osculating linear complexes of the asymptotic curves were first defined and discussed in detail from the standpoint of projective geometry by Professor Wilczynski.[*] If we denote by C' and C'' the osculating linear complexes of the asymptotic curves $v=\text{const.}$ and $u=\text{const.}$, respectively, he found the equations of these complexes to be

$$-b_v\,\bar{\omega}_{34}-b\,\bar{\omega}_{14}+b\,\bar{\omega}_{23}=0 \tag{15}$$

and

$$-a'_u\,\bar{\omega}_{42}+a'\,\bar{\omega}_{14}+a'\,\bar{\omega}_{23}=0, \tag{16}$$

respectively, where the quantities $\bar{\omega}_{ik}$, the Plückerian homogeneous line coordinates, are connected with the homogeneous point coordinates of the system $P_y\,P_z\,P_\rho\,P_\sigma$ in the usual way. We wish to investigate some of the metrical properties of these complexes and shall use the special surface coordinate system which has been defined.

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[*] *Transactions*, Vol. IX, p. 90, *et seq.*

From equations (14) the transformations for the line coordinates are found to be

$$\left.\begin{aligned} \overline{\omega}_{34} &= -\alpha HD'\sqrt{E}\omega_{31}+\beta HD'\sqrt{E}\omega_{23}, \\ \overline{\omega}_{42} &= \alpha HD'\sqrt{G}\omega_{31}+\beta HD'\sqrt{G}\omega_{23}, \\ \overline{\omega}_{14} &= \alpha HD'(\alpha\sqrt{G}+b\sqrt{E})\omega_{31}+\beta HD'(a\sqrt{G}-b'\sqrt{E})\omega_{23}-H^2D'\omega_{34}, \\ \overline{\omega}_{23} &= HD'^2\omega_{12}-\alpha HD'[(b'-2c)\sqrt{E}-(a-2d)\sqrt{G}]\omega_{31} \\ &\qquad\qquad +\beta HD'[(b'-2c)\sqrt{E}+(a-2d)\sqrt{G}]\omega_{23}, \end{aligned}\right\} \quad (17)$$

where the coordinates ω_{ik} are defined by the relations

$$\left.\begin{aligned} &\omega_{14}=\xi_1-\xi_2, &&\omega_{24}=\eta_1-\eta_2, &&\omega_{34}=\zeta_1-\zeta_2, \\ &\omega_{23}=\eta_1\zeta_2-\eta_2\zeta_1, &&\omega_{31}=\xi_2\zeta_1-\xi_1\zeta_2, &&\omega_{12}=\xi_1\eta_2-\xi_2\eta_1; \end{aligned}\right\} \quad (18)$$

ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 being two points of the line.

By transformations (17) the equation of complex C', equation (15), becomes

$$bD'\omega_{12}-\alpha M_1\omega_{31}+\beta M_2\omega_{23}+bH\omega_{34}=0, \quad (19)$$

where

$$M_1=M'\sqrt{E}+2bd\sqrt{G},\quad M_2=M'\sqrt{E}-2bd\sqrt{G},\quad M'=2bb'-2bc-b_v. \quad (20)$$

If we use the quantities ω_{ik} as given in (18), the equation of any linear complex may be written in the form

$$a_{12}\omega_{12}+a_{31}\omega_{31}+a_{14}\omega_{14}+a_{23}\omega_{23}+a_{34}\omega_{34}+a_{24}\omega_{24}=0,$$

where we have used the homogeneous notation for the coefficients also. The equations of the axis of the complex will then be*

$$\frac{\xi-\dfrac{a_{24}a_{12}-a_{34}a_{31}}{a_{23}^2+a_{31}^2+a_{12}^2}}{a_{23}}=\frac{\eta-\dfrac{a_{34}a_{23}-a_{14}a_{12}}{a_{23}^2+a_{31}^2+a_{12}^2}}{a_{31}}=\frac{\zeta-\dfrac{a_{14}a_{31}-a_{24}a_{23}}{a_{23}^2+a_{31}^2+a_{12}^2}}{a_{12}}. \quad (21)$$

Therefore, from (21) the equations of the axis of the complex C' are

$$\frac{\xi-(\alpha bHM_1/V'^2)}{\beta M_2}=\frac{\eta-(\beta bHM_2/V'^2)}{-\alpha M_1}=\frac{\zeta}{bD'}, \quad (22)$$

where

$$V'^2=\alpha^2M_1^2+\beta^2M_2^2+b^2D'^2=M'^2E-4bdFM'+4b^2d^2G+b^2D'^2. \quad (23)$$

We shall call the plane through the surface point perpendicular to the axis of the complex the *principal plane*, and its intersection with the tangent plane the *principal line*. The equation of the principal plane with respect to the complex C' is found to be

$$\beta M_2\xi-\alpha M_1\eta+bD'\zeta=0, \quad (24)$$

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* Plücker, "Neue Geometrie des Raumes," p. 32.

and, therefore, the equations of the principal line are

$$\beta M_2 \xi - \alpha M_1 \eta = 0, \quad \zeta = 0. \tag{25}$$

The point on the principal line where the axis of the complex intersects the tangent plane is at a distance δ' from the surface point, where

$$\delta' = bHU'/V'^2 \text{ and } U'^2 = \alpha^2 M_1^2 + \beta^2 M_2^2. \tag{26}$$

In order to find the *parameter* of the complex we change from the surface trihedral to another rectangular system, ξ', η', ζ', whose origin is the point of intersection of the axis of the complex with the tangent plane, the ξ' axis coinciding with the principal line and the ζ' axis with the axis of the complex.

From equations (25) and (22) the direction cosines of the ξ' and ζ' axes are

$$\alpha M_1/U',\ \beta M_2/U',\ 0, \text{ and } \beta M_2/V',\ -\alpha M_1/V',\ bD'/V',$$

respectively. Consequently, the direction cosines of the η' axis are

$$-\beta bD' M_2/U'V',\ \alpha bD' M_1/U'V',\ U'/V',$$

the matter of the positive direction of the lines being left arbitrary. Therefore, the equations of transformation from the surface trihedral to the special coordinate system just defined are

$$\begin{aligned}\xi &= (\alpha M_1/U')\xi' - (\beta bD'M_2/U'V')\eta' + (\beta M_2/V')\zeta' + (\alpha bHM_1/V'^2),\\ \eta &= (\beta M_2/U')\xi' + (\alpha bD'M_1/U'V')\eta' - (\alpha M_1/V')\zeta' + (\beta bHM_2/V'^2),\\ \zeta &= (U'/V')\eta' + (bD'/V')\zeta'.\end{aligned}$$

The corresponding transformations for the line coordinates are found to be

$$\left.\begin{aligned}\omega_{12} &= (bD'/V')\omega'_{12} + (U'/V')\omega'_{31} - (b^2HD'U'/V'^3)\omega'_{24} + (bHU'^2/V'^3)\omega'_{34},\\ \omega_{23} &= (\beta M_2/V')\omega'_{12} - (\beta bD'M_2/U'V')\omega'_{31} + (\alpha M_1/U')\omega'_{23}\\ &\qquad - (\beta bHM_2U'/V'^3)\omega'_{24} - (\beta b^2HD'M_2/V'^3)\omega'_{34},\\ \omega_{34} &= (U'/V')\omega'_{24} + (bD'/V')\omega'_{34},\\ \omega_{31} &= -(\alpha M_1/V')\omega'_{12} + (\alpha bD'M_1/U'V')\omega_{31} + (\beta M_2/U')\omega'_{23}\\ &\qquad + (\alpha bHM_1U'/V'^3)\omega'_{24} + (\alpha b^2HD'M_1/V'^3)\omega'_{34}.\end{aligned}\right\} \tag{27}$$

Substituting the values (27) in equation (19) we have as the transformed equation of the complex the equation

$$\omega'_{12} + (b^2HD'/V'^2)\omega'_{34} = 0.$$

Therefore, the parameter of complex C' is

$$P' = b^2HD'/V'^2. \tag{28}$$

We now consider the corresponding properties of the complex C''. The equation of this complex referred to the surface trihedral is

$$a' D' \omega_{12} + \alpha N_1 \omega_{31} + \beta N_2 \omega_{23} - a' H \omega_{34} = 0, \tag{29}$$

where

$$N_1 = N' \sqrt{G} + 2 a' c \sqrt{E}, \quad N_2 = N' \sqrt{G} - 2 a' c \sqrt{E}, \quad N' = 2 a a' - 2 a' d - a'_u, \tag{30}$$

and the equations of its axis are

$$\frac{\xi - (\alpha a' H N_1 / V''^2)}{\beta N_2} = \frac{\eta + (\beta a' H N_2 / V''^2)}{\alpha N_1} = \frac{\zeta}{a' D'}, \tag{31}$$

where

$$V''^2 = \alpha^2 N_1^2 + \beta^2 N_2^2 + a'^2 D'^2 = N'^2 G - 4 a' c F N' + 4 a'^2 c^2 E + a'^2 D'^2. \tag{32}$$

The principal plane with respect to the complex C'' is given by

$$\beta N_2 \xi + \alpha N_1 \eta + a' D' \zeta = 0,$$

and, therefore, the principal line with respect to C'' has the equations

$$\beta N_2 \xi + \alpha N_1 \eta = 0, \quad \zeta = 0.$$

The distance from the surface point to the point in which the axis of this complex intersects the tangent plane is given by

$$\delta'' = a' H U'' / V''^2, \text{ where } U''^2 = \alpha^2 N_1^2 + \beta^2 N_2^2. \tag{33}$$

We make a similar change of coordinate system in order to find the parameter of complex C'' as in the case of complex C'. The transformed equation of the complex is

$$\omega''_{12} - (a'^2 H D' / V''^2) \omega''_{34} = 0,$$

so that the parameter has the value

$$P'' = -a'^2 H D' / V''^2. \tag{34}$$

Comparing the parameters P' and P'' of the two complexes we see that their values are opposite in sign, if the surface and the two asymptotic curves concerned are real. Therefore, *the linear complexes which osculate two real asymptotic curves of a real surface at their point of intersection are oppositely twisted.*

If ϕ is the angle between the axes of the complexes C' and C'' and if K denotes the perpendicular distance between them, we find from equations (22) and (31) that these quantities are given by

$$\cos \phi = Q^2 / V' V'', \tag{35}$$

$$K = H D' (a'^2 V'^2 - b^2 V''^2) Q^2 / V'^2 V''^2 \sqrt{V'^2 V''^2 - Q^4}, \tag{36}$$

where

$$Q^2 = \beta^2 M_2 N_2 - \alpha^2 M_1 N_1 + b a' D'^2. \tag{37}$$

IV. THE PENCIL OF LINEAR COMPLEXES DETERMINED BY THE COMPLEXES C' AND C''.

The equation of any complex C_λ of the pencil determined by C' and C'' is a linear combination of equations (19) and (29) and is, therefore, of the form

$$D'(b+\lambda a')\omega_{12}-\alpha(M_1-\lambda N_1)\omega_{31}+\beta(M_2+\lambda N_2)\omega_{23}+H(b-\lambda a')\omega_{34}=0. \quad (38)$$

The equation of the principal plane with respect to any complex C_λ of the pencil is

$$\beta(M_2+\lambda N_2)\xi-\alpha(M_1-\lambda N_1)\eta+D'(b+\lambda a')\zeta=0,$$

and, therefore, the equations of the principal line with respect to C_λ are

$$\beta(M_2+\lambda N_2)\xi-\alpha(M_1-\lambda N_1)\eta=0, \quad \zeta=0. \quad (39)$$

The line conjugate to the surface normal with respect to C_λ is given by the equations

$$\alpha(M_1-\lambda N_1)\xi+\beta(M_2+\lambda N_2)\eta=0, \quad \zeta=0. \quad (40)$$

All of these lines form a pencil in the tangent plane whose vertex is the point

$$\xi=\beta P_2/A, \quad \eta=-\alpha Q_2/A, \quad \zeta=0,$$

where we have put

$$A=M'N'-4ba'cd, \quad (41)$$

and

$$\left.\begin{aligned} P_2&=a'\sqrt{E}(M'-2bc)+b\sqrt{G}(N'-2a'd), \\ Q_2&=a'\sqrt{E}(M'-2bc)-b\sqrt{G}(N'-2a'd). \end{aligned}\right\} \quad (42)$$

Corresponding lines of the pencils given by equations (39) and (40) are perpendicular to each other. Therefore, the locus of their points of intersection is a circle. Its equations are

$$\xi^2+\eta^2-(\beta P_2/A)\xi+(\alpha Q_2/A)\eta=0, \quad \zeta=0.$$

We may state the theorem:

The line which corresponds to the surface normal in any one of the complexes C_λ of the pencil is in the tangent plane of the corresponding surface point and perpendicular to the principal line with respect to that complex.

The special complexes of the pencil (38) are given by the values of λ which satisfy the equation of the form

$$a_{12}a_{34}+a_{31}a_{42}+a_{14}a_{23}=0.$$

These values are $-b/a'$ and b/a'. We shall call the corresponding special complexes the special complexes of the first and second kind, respectively.

If we put $\lambda = -b/a'$ in equation (38) we obtain the equation of the special complex of the first kind. It is

$$\alpha P_1 \omega_{31} - \beta Q_1 \omega_{23} - 2ba'H\omega_{34} = 0, \tag{43}$$

where

$$\left.\begin{aligned} P_1 &= a'\sqrt{E}(M'+2bc) + b\sqrt{G}(N'+2a'd), \\ Q_1 &= a'\sqrt{E}(M'+2bc) - b\sqrt{G}(N'+2a'd). \end{aligned}\right\} \tag{44}$$

The equations of the axis of this complex, or of the directrix of the first kind, are found to be

$$\frac{\xi - (2\alpha ba'HP_1/V_1^2)}{\beta Q_1} = \frac{\eta - (2\beta ba'HQ_1/V_1^2)}{-\alpha P_1} = \frac{\zeta}{0}, \tag{45}$$

where

$$V_1^2 = \alpha^2 P_1^2 + \beta^2 Q_1^2 = a'^2 V'^2 + b^2 V''^2 - 2ba'Q^2. \tag{46}$$

The equation of the special complex of the second kind is

$$2ba'D'\omega_{12} - \alpha Q_2 \omega_{31} + \beta P_2 \omega_{23} = 0, \tag{47}$$

and the equations of its axis, or of the directrix of the second kind, are

$$\xi/\beta P_2 = \eta/-\alpha Q_2 = \zeta/2ba'D', \tag{48}$$

in connection with which equations we introduce the quantity V_2 defined by

$$V_2^2 = \alpha^2 Q_2^2 + \beta^2 P_2^2 + 4b^2a'^2D'^2 = a'^2V'^2 + b^2V''^2 + 2ba'Q^2. \tag{49}$$

Equations (45) and (48) enable us to compute a number of important quantities. Let ψ be the angle between the two directrices, I the perpendicular distance between them, l, m, n the direction cosines of their common perpendicular, $\underline{\xi}, \underline{\eta}, \underline{\zeta}$ the coordinates of the middle point of the congruence. Then we shall have

$$\left.\begin{gathered} \cos\psi = (a'^2V'^2 - b^2V''^2)/V_1V_2, \quad I = 2ba'HD'/V_4, \\ l = \alpha D'P_1/V_4, \quad m = \beta D'Q_1/V_4, \quad n = -HA/V_4, \\ \underline{\xi} = H(\alpha ba'D'^2P_1 + \beta HAP_2)/V_4^2, \quad \underline{\eta} = H(\beta ba'D'^2Q_1 - \alpha HAQ_2)/V_4^2, \\ \underline{\zeta} = ba'H^2D'A/V_4^2, \end{gathered}\right\} \tag{50}$$

where

$$V_4^2 = V_1^2D'^2 + H^2A^2.$$

V. THE CYLINDROID FORMED BY THE AXES OF THE LINEAR COMPLEXES OF THE PENCIL DETERMINED BY THE COMPLEXES C' AND C''.

The ruled surface formed by the axes of a pencil of linear complexes is a special ruled surface of the third degree, usually known as a cylindroid. In order to obtain a simple form for the equation of the cylindroid, which belongs to the pencil of linear complexes C_λ, we introduce a new rectangular system of coordinates, whose origin is the middle point of the congruence, whose ζ''' axis is the common perpendicular of the two directrices, and whose ξ''' and η''' axes bisect the angles made by the lines parallel to the directrices through the origin.

From equations (45) and (48) we find the direction cosines of the ξ''' and η''' axes to be

$\beta S_1/W_1,\ -\alpha R_1/W_1,\ 2ba'D'V_1/W_1$ and $\beta S_2/W_2,\ -\alpha R_2/W_2,\ -2ba'D'V_1/W_2$,

respectively, where

$$\begin{aligned}
&V_3^2 = \alpha^2 P_1 Q_2 + \beta^2 P_2 Q_1,\\
&S_1 = Q_1 V_2 + P_2 V_1, \quad S_2 = Q_1 V_2 - P_2 V_1,\\
&R_1 = P_1 V_2 + Q_2 V_1, \quad R_2 = P_1 V_2 - Q_2 V_1,\\
&W_1^2 = 2 V_1 V_2 (V_1 V_2 + V_3^2), \quad W_2^2 = 2 V_1 V_2 (V_1 V_2 - V_3^2).
\end{aligned}$$

The direction cosines of the ζ''' axis were given among the relations (50). The positive directions in regard to these new coordinate axes are left arbitrary. Then the necessary transformations of the point coordinates from the surface trihedral to this new system of coordinates are given by

$$\begin{aligned}
\xi &= (\beta S_1/W_1)\xi''' + (\beta S_2/W_2)\eta''' + (\alpha D'P_1/V_4)\zeta''' + H(\alpha ba'D'^2P_1 + \beta HAP_2)/V_4^2,\\
\eta &= (-\alpha R_1/W_1)\xi''' - (\alpha R_2/W_2)\eta''' + (\beta D'Q_1/V_4)\zeta''' + H(\alpha ba'D'^2Q_1 - \beta HAQ_2)/V_4^2,\\
\zeta &= (2ba'D'V_1/W_1)\xi''' - (2ba'D'V_1/W_2)\eta''' - (HA/V_4)\zeta''' + ba'H^2D'A/V_4^2.
\end{aligned}$$

We shall not write down the corresponding transformations of the line coordinates.

We find the equations of the complexes C' and C'', referred to this new system of coordinates, to be

$$\begin{aligned}
&[ba'D'H(V_2+V_1)/W_1]\omega_{14}''' + [ba'D'H(V_2-V_1)/W_2]\omega_{24}'''\\
&\qquad + [(V_2+V_1)V_4/W_2]\omega_{23}''' - [(V_2-V_1)V_4/W_1]\omega_{31}''' = 0, \qquad (51)
\end{aligned}$$

and

$$\begin{aligned}
&[ba'D'H(V_2-V_1)/W_1]\omega_{14}''' + [ba'D'H(V_2+V_1)/W_2]\omega_{24}'''\\
&\qquad + [(V_2-V_1)V_4/W_2]\omega_{23}''' - [(V_2+V_1)V_4/W_1]\omega_{31}''' = 0, \qquad (52)
\end{aligned}$$

respectively.

According to the general theory of a pencil of linear complexes, when the equations of the fundamental complexes are in the form of equations (51) and (52), the equation of the cylindroid formed by the axes of the complexes of the pencil is of the form*

$$(\xi'''^2 + \eta'''^2)\zeta''' = (a_{14}/a_{23} - a_{24}/a_{31})\xi'''\eta''',$$

a_{14}, a_{23}, a_{24}, a_{31} being taken from the equation of either of the fundamental complexes. We find from either equation (51) or (52) that

$$\frac{a_{14}}{a_{23}} = ba'D'HW_2/V_4W_1 \quad \text{and} \quad \frac{a_{24}}{a_{31}} = -ba'D'HW_1/V_4W_2.$$

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* Plücker, "Neue Geometrie des Raumes," p. 97.

Therefore, the equation of the cylindroid in this case is

$$(\xi'''^2+\eta'''^2)\zeta''' = (4ba'D'HV_1^2V_2^2/W_1W_2V_4)\xi'''\eta'''. \tag{53}$$

If we let

$$T = 2ba'D'HV_1^2V_2^2/W_1W_2V_4,$$

then the cylindroid lies between the two planes

$$\zeta'''=T \quad \text{and} \quad \zeta'''=-T;$$

that is, these planes are the tangent planes at the pinch points,

$$\xi'''=0,\ \eta'''=0,\ \zeta'''=T;\quad \xi'''=0,\ \eta'''=0,\ \zeta'''=-T.$$

VI. THE DIRECTRIX CURVES.

Professor Wilczynski has shown* that there exist, in general, two one-parameter families of curves on the surface S such that, if the point P describes one of these curves, both of its directrices will simultaneously describe developable surfaces. He called these curves *directrix curves.* The quadratic equation which determines the tangents of the two directrix curves which pass through the point P he found to be†

$$bL\delta u^2+2M\delta u\delta v-a'N\delta v^2=0,$$

where

$$\left.\begin{aligned} L &= -2a'(2ba'f+2abb_v+ba'_{uu})+ba_u'^2, \\ M &= 2ba'(a'b_{uv}-ba'_{uv})+2b^2a'_ua'_v-2a'^2b_ub_v, \\ N &= -2b(2ba'g+2ba'a'_u+a'b_{vv})+a'b_v^2, \end{aligned}\right\} \tag{54}$$

the surface S having been defined by equations (3). We wish to find the corresponding equation when the surface is defined by equations (1).

Equations (1) may be changed into equations (3) by a transformation of the form $y=\lambda\bar{y}$. Then the quantities f and g in equations (3) have the values‡

$$f=c-a_u-a^2-2bb',\quad g=c'-b'_v-b'^2-2aa'.$$

If in these expressions we make $c=c'=0$ and substitute the resulting values of f and g in the quantities (54), the corresponding equation of the directrix curves can be put in the form

$$\begin{aligned} b^2(C^2+2a'C_u+4a'^2B)\delta u^2-2[a'^2(bB_u-Bb_u)-b^2(a'C_v-Ca'_v)]\delta u\delta v \\ -a'^2(B^2+2bB_v+4b^2C)\delta v^2=0, \end{aligned} \tag{55}$$

in which the quantities B and C have the values

$$B=M'+2bc,\quad C=N'+2a'd.$$

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* *Transactions,* Vol. IX, p. 114, *et seq.*

† *Transactions,* Vol. IX, p. 116.

‡ *Transactions,* Vol. VIII, p. 246.

VII. SPECIAL POINTS.

Since the pencil of linear complexes formed from the osculating linear complexes of the asymptotic lines plays an important part in the study made in this paper, and since these complexes are indeterminate if a' or b is equal to zero, it is clear that we have tacitly assumed that neither a' nor b shall vanish. Therefore, if the surface contains straight lines, all points upon such straight lines are to be excluded and, of course, all ruled surfaces are also excluded from consideration.

We wish to discuss the values of K, $\cos\phi$ and $\cos\psi$ in (36), (35) and (50), respectively. These quantities are seen to be indeterminate in value when either a' or b is zero. If there are other values of the parameters different from those that make either a' or b zero, for which these quantities become indeterminate, the corresponding points must also be excluded. Thus, the developments of this chapter apply only to such points for which the quantity Q^2 or $a'^2V'^2-b^2V''^2$ is equal to zero while neither V', nor V'' nor $V'^2V''^2-Q^4$ is equal to zero.

a) *Points for Which the Axes of the Complexes C' and C'' Intersect.*

The axes of these complexes intersect when K is zero. This quantity is zero under the two separate conditions,

$$Q^2=0 \tag{56}$$

and

$$a'^2V'^2-b^2V''^2=0. \tag{57}$$

We observe that equation (56) makes the quantity $\cos\phi$ also zero and that $\cos\psi$ is zero under the condition (57). Therefore, we can state the following theorems:

If the axes of the osculating linear complexes of the asymptotic lines are determinate and intersect, either these axes are perpendicular to each other or the directrices of the first and second kind are perpendicular to each other, or both conditions may be fulfilled at the same time.

If the axes of the osculating linear complexes of the asymptotic lines are determinate and perpendicular to each other, they intersect.

If the directrices of the first and second kind are perpendicular to each other, the axes of the osculating linear complexes of the asymptotic lines intersect if they are determinate.

b) *Points for Which the Directrices of the First and Second Kind Are Perpendicular to Each Other.*

Such points were defined by equation (57). They are of particular interest on account of the further specialization which exists in the geometrical configuration to which the directrices belong and whose metrical properties we have developed.

If we consider the condition that the parameters P' and P'' of the complexes C' and C'' be numerically equal, we find from (28) and (34) that it is exactly equation (57). We have, therefore, the theorem:

If the asymptotic tangents of a point of a surface are real and if its directrices are perpendicular to each other, the parameters of the osculating linear complexes of the corresponding asymptotic lines are numerically equal and opposite in sign, and conversely.

The equation of the cylindroid reduces to

$$(\xi'''^2+\eta'''^2)\zeta''' = (2ba'D'/V_4)\xi'''\eta'''.$$

We see that the directrices are in this case the singular tangents to the cylindroid. Therefore, *when the directrices of the first and second kind are perpendicular to each other, they are the singular tangents to the cylindroid.*

Since the directrix of the first kind is in the tangent plane and in this case the two directrices are perpendicular to each other, the plane of the directrix of the second kind and the axis of the cylindroid must be perpendicular to the tangent plane. It is found to bisect the angle the axes of the complexes C' and C'' make with each other. Consequently, these axes must meet the tangent plane in points which are at the same distance from the surface point; that is, δ' and δ'' in (26) and (33), respectively, must be equal, which fact may easily be verified. We may state the theorem:

When the axes of the complexes C' and C'' intersect each other and the directrices of the first and second kind are perpendicular to each other, the axes make the same angle with the tangent plane and meet it in points equidistant from the surface point.

Let us write equation (57) in the form

$$(a'^2M'^2E-b^2N'^2G)-4ba'F(a'dM'-bcN')+4a'^2b^2(d^2G-c^2E)=0.$$

Evidently, it may be satisfied by means of the three separate equations

$$\left.\begin{array}{c}(a'M'\sqrt{E}-bN'\sqrt{G})(a'M'\sqrt{E}+bN'\sqrt{G})=0, \quad a'dM'-bcN'=0,\\ (d\sqrt{G}-c\sqrt{E})(d\sqrt{G}+c\sqrt{E})=0.\end{array}\right\} \qquad (58)$$

These may be divided into two sets, each of three separate equations, any two of which imply the third. They are

$$a'M'\sqrt{E}-bN'\sqrt{G}=0, \quad d\sqrt{G}-c\sqrt{E}=0, \quad a'dM'-bcN'=0, \tag{59}$$

and

$$a'M'\sqrt{E}+bN'\sqrt{G}=0, \quad d\sqrt{G}+c\sqrt{E}=0, \quad a'dM'-bcN'=0. \tag{60}$$

We observe that in the case given by equations (59) the directrix of the first kind is perpendicular to one of the principal normal planes, that consequently the directrix of the second kind and the axis of the cylindroid are in this plane, and that in the case of equations (60) a similar position exists with reference to the other principal normal plane.

c) *Points for Which the Directrix of the Second Kind Coincides with the Normal to the Surface.*

According to equation (48) the directrix of the second kind coincides with the normal to the surface if and only if P_2 and Q_2 separately are equal to zero; that is, if and only if

$$M'=2bc, \quad N'=2a'd. \tag{61}$$

Since for these points the directrices must be perpendicular to each other, these points must be included among the special points which were defined by equation (57). The fact that equation (57) is satisfied by equations (61) may easily be verified.

If in addition to equations (61) the equation

$$d\sqrt{G}-c\sqrt{E}=0$$

is satisfied, the directrix of the first kind is perpendicular to one of the principal normal planes, while if the additional equation which is satisfied is

$$d\sqrt{G}+c\sqrt{E}=0,$$

the same fact holds in regard to the other principal normal plane.

If the relations (61) be substituted in the quantities B and C which occur in equation (55), that equation of the directrix curves reduces to the form

$$E\delta u^2-G\delta v^2=0,$$

so that at these points the directrix curves coincide with the lines of curvature, as they obviously should.

By introducing in equations (61) the values of M' and N' from (20) and (30), respectively, we may write them in the form

$$b_v-2bb'+4bc=0, \quad a'_u-2aa'+4a'd=0. \tag{62}$$

If in the two general relations

$$\frac{H_u}{H}=\left\{\begin{matrix}11\\1\end{matrix}\right\}+\left\{\begin{matrix}12\\2\end{matrix}\right\},\quad \frac{H_v}{H}=\left\{\begin{matrix}12\\1\end{matrix}\right\}+\left\{\begin{matrix}22\\2\end{matrix}\right\},$$

we replace the Christoffel symbols by their values in a, b, c, d we obtain

$$c=-b'-H_v/2H,\quad d=-a-H_u/2H. \tag{63}$$

By means of these relations then, equations (62) become

$$Hb_v-2bH_v-6Hbb'=0,\quad Ha'_u-2a'H_u-6Haa'=0.$$

The conditions developed in this chapter characterize certain special kinds of special points on a surface if they are merely satisfied by special values of the parameters. They characterize special classes of surfaces if they can be satisfied identically. The existence, in general, of real points or surfaces of these several kinds will not be discussed in this paper. In the next chapter we shall show however by means of a particular surface that points of the several different kinds which have been defined actually exist.

VIII. THE MINIMAL SURFACE OF ENNEPER.

In order that the significance of the preceding developments may appear more clearly we proceed to apply them to a well-known surface, the Minimal Surface of Enneper. We shall obtain, in this way, some new properties of this surface.

The Minimal Surface of Enneper is represented by the equations

$$x=\tfrac{1}{4}(u+v)(u^2-4uv+v^2+6),\quad y=\tfrac{1}{4}(u-v)(u^2+4uv+v^2+6),\quad z=3uv, \tag{64}$$

where the curves $u=$const. and $v=$const. are the asymptotic lines, the lines of curvature being given by the equations $u+v=$const. and $u-v=$const.

The fundamental quantities E, F, G, H, D' and the quantities α, β, defined in (12), are found to have the values,

$$E=\frac{9}{8}W^2,\quad F=0,\quad G=\frac{9}{8}W^2,\quad H=\frac{9}{8}W^2,\quad D'=-3,\quad \alpha=\frac{1}{\sqrt{2}},\quad \beta=\frac{1}{\sqrt{2}},$$

where

$$W=u^2+v^2+2.$$

The quantities a, a', b, b' c, d which occur in (6) and (10) are,

$$a=-u/W,\quad a'=u/W,\quad b=v/W,\quad b'=-v/W,\quad c=-v/W,\quad d=-u/W,$$

We see from the values just given that the condition that neither a' nor b shall be zero implies for this surface that neither u nor v shall be equal to zero. The zero values of the parameters give the intersection of the surface with the XY-plane, which is made up of the two straight lines, $x+y=0$, $x-y=0$.

Therefore, every point upon these two straight lines is excluded from our considerations.

By easy substitutions we obtain the following equations and results in connection with the complexes C' and C'', the surface coordinate system being used:

Equation of complex C',

$$8v\omega_{12}-2(W-2v^2+2uv)\omega_{31}+2(W-2v^2-2uv)\omega_{23}-3vW^2\omega_{34}=0;$$

Equation of complex C'',

$$8u\omega_{12}+2(W-2u^2+2uv)\omega_{31}+2(W-2u^2-2uv)\omega_{23}+3uW^2\omega_{34}=0;$$

Equations of axis of complex C',

$$\frac{\xi+\frac{3}{4}v[W+2v(u-v)]}{-[W-2v(u+v)]}=\frac{\eta+\frac{3}{4}v[W-2v(u+v)]}{[W+2v(u-v)]}=\frac{\zeta}{4v}; \tag{65}$$

Equations of axis of complex C'',

$$\frac{\xi+\frac{3}{4}u[W-2u(u-v)]}{[W-2u(u+v)]}=\frac{\eta-\frac{3}{4}u[W-2u(u+v)]}{[W-2u(u-v)]}=\frac{\zeta}{4u}; \tag{66}$$

The parameters P' and P'' are given by

$$P'=-3v^2, \quad P''=3u^2.$$

We find that the expression Q^2 in (56) is identically zero except when the parameters have the value zero and that each of the two quantities V' and V'' has the value 9/8. No points then are excluded by the limitation that was placed upon the quantities V', V'' and Q^2. Therefore, every general point on the surface is of the type defined by equation (56), that is, for every such point the axes of the osculating linear complexes of the asymptotic lines intersect and are perpendicular to each other.

The equations of the special complexes of the first and second kind and of the two directrices are as follows:

Equation of special complex of first kind,

$$(u+v)\omega_{31}-(u-v)\omega_{23}+3uvW\omega_{34}=0;$$

Equation of special complex of second kind,

$$8uv\omega_{12}-(u-v)(W+4uv)\omega_{31}+(u+v)(W-4uv)\omega_{23}=0;$$

Equations of directrix of first kind,

$$\frac{\xi+(3uv(u+v)W/2(u^2+v^2))}{u-v}=\frac{\eta+(3uv(u-v)W/2(u^2+v^2))}{-(u+v)}=\frac{\zeta}{0}; \tag{67}$$

Equations of directrix of second kind,

$$\frac{\xi}{(u+v)(W-4uv)}=\frac{\eta}{-(u-v)(W+4uv)}=\frac{\zeta}{8uv}. \tag{68}$$

The expressions for the several quantities in (50) and (54) are given by,

$$\left.\begin{array}{l}\cos\psi=(u-v)(u+v)/(u^2+v^2),\quad I=6uv,\quad l=2(u+v)/W,\\ m=2(u-v)/W,\quad \eta=(W-4)/W,\\ \underline{\xi}=3(u+v)[(u-v)^4-4(u^2v^2+1)]/4W,\\ \underline{\eta}=-3(u-v)[(u+v)^4-4(u^2v^2+1)]/4W,\\ \underline{\zeta}=3uv(W-4)/W,\quad T=-3(u^2+v^2)/2.\end{array}\right\} \tag{69}$$

Equation (57) here assumes the form

$$(u-v)(u+v)=0.$$

Therefore, the directrices are perpendicular to each other at the points given by the two conditions, $u=v\neq 0$ and $u=-v\neq 0$. But the condition $u=v\neq 0$ satisfies equations (59) for this surface and the condition $u=-v\neq 0$ equations (60). Therefore, the points given by these two conditions are of the more special types defined by equations (59) and (60).

The curves defined on the surface by the two conditions $u=v\neq 0$ and $u=-v\neq 0$ are found from (64) to be given by

$$x^2=\frac{z}{27}(9-z)^2,\quad y=0, \tag{70}$$

and

$$x=0,\quad y^2=\frac{-z}{27}(9+z)^2, \tag{71}$$

respectively. Every point of these two plane cubic curves, except the origin, is of the special kind described.

For this surface the two equations (61) reduce to

$$u^2-3v^2+2=0,\quad 3u^2-v^2-2=0,$$

which have the solutions $u=\pm 1$ and $v=\pm 1$. These values of the parameters give four points of this very special type, for which the directrix of the second kind coincides with the normal to the surface. The points are

$$\begin{array}{llll} x=2, & x=-2, & x=0, & x=0,\\ y=0, & y=0, & y=2, & y=-2,\\ z=3, & z=3, & z=-3, & z=-3.\end{array}$$

Of course these points are on the curves given in (70) and (71) and are found to be the maximum and minimum points on the ovals of those curves with reference to their axes of symmetry.

Equation (55), the differential equation of the directrix curves, becomes $\delta u^2/u^2 - \delta v^2/v^2 = 0$, so that these curves are given by $uv = c_1$ and $u/v = c_2$. They may be represented parametrically by

$$
\begin{aligned}
x &= (u^2+c_1)\,[u^4+(6-4c_1)\,u^2+c_1^2]/4u^3,\\
y &= (u^2-c_1)\,[u^4+(6+4c_1)\,u^2+c_1^2]/4u^3,\\
z &= 3c_1,
\end{aligned}
$$

and

$$
\begin{aligned}
x &= u\,(1+c_2)\,[u^2(1-4c_2+c_2^2)+6c_2^2]/4c_2^3,\\
y &= u\,(1-c_2)\,[u^2(1+4c_2+c_2^2)+6c_2^2]/4c_2^3,\\
z &= 3u^2/c_2,
\end{aligned}
$$

respectively.

We observe that the directrix curve directions are conjugate, that along a curve given by $uv = c_1$, a plane curve, the two directrices are a constant distance apart and that for the points on the two curves (70) and (71) the directrix curve directions are the same as those of the lines of curvature.

It has been found that in general the axes of the two complexes C' and C'', whose equations were given in (72) and (73) intersect. Their point of intersection proves to be the middle point of the congruence, the point ξ, η, ζ given in (69), which we will call the point C.

The two directrices whose equations were given in (67) and (68) intersect the axis of the cylindroid, the line which goes through the point ξ, η, ζ and has the direction cosines l, m, n, as given in (69), in the points C_1 and C_2 which are as follows:

Point C_1,

$$\xi = \tfrac{3}{4}(u+v)\,[W-4(uv+1)], \quad \eta = -\tfrac{3}{4}(u-v)\,[W+4(uv-1)], \quad \zeta = 0,$$

Point C_2,

$$
\begin{aligned}
\xi &= 3\,(u+v)\,(W-4)\,(W-4uv)/4W,\\
\eta &= -3\,(u-v)\,(W-4)\,(W+4uv)/4W,\\
\zeta &= 6uv\,(W-4)/W.
\end{aligned}
$$

As the point P traces the surface each of the three points C, C_1, C_2 traces a surface which is covariant with the original surface.

From equations (13) the equations of transformation from the surface trihedral to the fixed trihedral, with reference to which the equations of the surface are given, are found to be

$$
\begin{aligned}
\bar{\xi} &= \frac{-2\,(uv-1)}{W}\,\xi - \frac{(u^2-v^2)}{W}\,\eta + \frac{2\,(u+v)}{W}\,\zeta + x,\\
\bar{\eta} &= \frac{(u^2-v^2)}{W}\,\xi - \frac{2\,(uv+1)}{W}\,\eta - \frac{2\,(u-v)}{W}\,\zeta + y,\\
\bar{\zeta} &= \frac{2\,(u+v)}{W}\,\xi + \frac{2\,(u-v)}{W}\,\eta + \frac{(W-4)}{W}\,\zeta + z.
\end{aligned}
$$

By means of these equations we find that the coordinates of the points C, C_1, C_2, referred to the fixed coordinate system, have the following form:

$$\begin{array}{llll} C. & x=u^3+v^3, & C_1. \quad x=u^3+v^3, & C_2. \quad x=u^3+v^3, \\ & y=u^3-v^3, & \qquad y=u^3-v^3, & \qquad y=u^3-v^3, \\ & z=0, & \qquad z=-3uv, & \qquad z=3uv. \end{array}$$

These expressions for the coordinates of these points show several important relations: That the axis of the cylindroid is always perpendicular to the XY-plane, and consequently the axes of the complexes and the directrices are always parallel to this plane; that the middle point of the congruence is in the XY-plane and, therefore, the axes of the complexes are in this plane; that the covariant surfaces of the points C_1 and C_2 are exactly like in character, being reciprocally symmetrical surfaces with references to the XY-plane.

The covariant surfaces of the points C_1 and C_2 are the simple cubic surfaces

$$x^2-y^2=-\frac{4}{27}z^3, \text{ and } x^2-y^2=\frac{4}{27}z^3,$$

respectively. These surfaces have in common the lines $x+y=0$ and $x-y=0$ in the XY-plane, lines which are upon the Surface of Enneper itself and the points of which were excluded from our considerations.

It is easy to show that if the point P moves along a line of curvature, an asymptotic line or a directrix curve of the Enneper Surface, the three points C, C_1, C_2, which correspond to P in every case trace a plane curve.

We have studied in this paper the metrical properties of certain geometrical configurations which have been built up in connection with a point on a surface. We have made a study of these configurations for all the points of the Surface of Enneper to which the developments apply and have found the following properties of this surface:

a) *The axis of the cylindroid is always perpendicular to the XY-plane, with the middle point of the congruence always in this plane. Consequently, the axes of the osculating linear complexes of the asymptotic lines and the two directrices are always parallel to the XY-plane.*

b) *The axes of these complexes intersect and are perpendicular to each other. Their point of intersection is the middle point of the congruence and consequently they are in the XY-plane.*

c) *The surface contains two plane curves, the intersections of the surface by the XZ- and YZ-planes, at all of whose points, excepting the origin, the* *directrices of the first and second kind are perpendicular to each other, the axis of the cylindroid in each case lying in the plane of the curve. For all these points the parameters of the complexes are numerically equal and opposite in sign.*

d) *There are four points for which the directrix of the second kind coincides with the normal to the surface. Two of these points are upon each of the curves in the XZ- and YZ-planes and are the maximum and minimum points on the oval of these curves with reference to the axes of symmetry.*

By means of the consideration of this special Surface of Enneper we have verified the results of the paper and have established the existence of real points of all of the special types that have been defined; points for which the axes of the osculating linear complexes of the asymptotic lines intersect and are perpendicular to each other, other points for which these axes intersect while the directrices of the two kinds are perpendicular to each other, and points for which the directrix of the second kind coincides with the normal to the surface. In the absence of any general existence theorems the fact that such points are shown to exist has an important bearing.

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